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Connes' model building kit

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Abstract

A. Connes and J. Lott's applications of non-commutative geometry to interaction physics are described for the purpose of model building.

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The possibilities of the Yang–Mills–Higgs model building kit have been explored thoroughly in the last 30 years and the standard $SU(3) \times SU(2) \times U(1)$ model of strong and electroweak interactions has emerged quite uniquely as adequate description of high energy phenomena. On the mathematical side this kit relies on two ingredients: the differential forms on spacetime together with their structure of differential algebra and a Lie group represented on a finite dimensional internal space. These two ingredients are thrown together in a tensor product to yield a gauge invariant action. In the Connes–Lott approach [1] both spacetime and internal space are described by involution algebras and their tensor product produces a special class of Yang–Mills actions with spontaneous symmetry breaking. In this class the fermionic mass matrix is naturally unified with the Dirac operator, the Higgs scalars with the gauge bosons and the Higgs potential with the Yang–Mills Lagrangian. While the introduction of Lie groups is ad hoc, involution algebras have a profound mathematical motivation in this context, non-commutative geometry. A model builder, however, who is willing to accept Connes and Lott's rules can very well do with a minimum of mathematics to be introduced below.

Let us quickly review input and output of the Yang–Mills–Higgs kit. To get started we have to commit ourselves to the following choices:
- a (finite dimensional) real, compact Lie group G,

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- a positive definite, bilinear invariant form on the Lie algebra $\mathfrak{g}$ of G. This choice is parametrized by a few positive numbers g_i, the coupling constants,
- a (unitary) representation $\mathcal{H}_L$ for left handed fermions (spin $\frac{1}{2}$),
- a representation $\mathcal{H}_R$ for right handed fermions,
- a representation $\mathcal{H}_S$ for scalars (spin 0),
- an invariant positive polynomial of order 4 on the representation space of the scalars; this polynomial is denoted by $V(\phi), \phi \in \mathcal{H}_S$, the Higgs potential,
- one complex number or Yukawa coupling g_Y for every singlet in the decomposition of the representation

$$\left(\mathcal{H}_L^* \otimes \mathcal{H}_R \otimes \mathcal{H}_S\right) \oplus \left(\mathcal{H}_R^* \otimes \mathcal{H}_L \otimes \mathcal{H}_S\right). \tag{1}$$

With these ingredients the popular Yang-Mills-Higgs algorithm produces the following output:

- the particle spectrum, roughly one particle for each basis vector in

$$\mathfrak{g}^{\mathbb{C}} \oplus \mathcal{H}_L \oplus \mathcal{H}_R \oplus \mathcal{H}_S; \tag{2}$$

  the basis elements of the complexified Lie algebra $\mathfrak{g}^{\mathbb{C}}$ are the gauge bosons, they have spin 1,
- the particle masses,
- the interactions, e.g. charges, Kobayashi-Maskawa matrix, etc.

Trial and error, that is guessing an input, calculating the output and comparing with experiment, has singled out the standard model of electroweak and strong interactions:

$$G = SU(3) \times SU(2) \times U(1) \tag{3}$$

with three coupling constants g_3, g_2, g_1,

$$\mathcal{H}_L = \left[(1,2,-1) \oplus (3,2,\tfrac{1}{3})\right] \times 3, \tag{4}$$

$$\mathcal{H}_R = \left[(1,1,-2) \oplus (3,1,\tfrac{4}{3}) \oplus (3,1,-\tfrac{2}{3})\right] \times 3, \tag{5}$$

$$\mathcal{H}_S = (1,2,-1), \tag{6}$$

where (n_3, n_2, y) denotes a tensor product of an n_3 dimensional representation under $SU(3)$, an n_2 dimensional representation under $SU(2)$ and the one dimensional representation of $U(1)$ with hypercharge y:

$$\rho(e^{i\theta}) = e^{i3y\theta}, \qquad 3y \in \mathbb{Z},\ \theta \in [0, 2\pi), \tag{7}$$

$$V(\phi) = \lambda(\phi^*\phi)^2 - \tfrac{1}{2}\mu^2\phi^*\phi, \qquad \phi \in \mathcal{H}_S \quad \lambda, \mu > 0. \tag{8}$$

There are 27 Yukawa couplings in the input, enough to allow for an arbitrary fermionic mass matrix (fermion masses and Kobayashi-Maskawa matrix) and an arbitrary scalar mass in the output.

In a first step we only consider a particular subset of Connes' model building kit, where spacetime and internal space come in a tensor product. This subset compares

naturally with the Yang-Mills-Higgs model building kit. As before the input concerns only the (finite dimensional) internal space:
- an associative involution algebra $\mathcal{A}$ with unit 1,
- two representations $\mathcal{H}_L$ and $\mathcal{H}_R$ of $\mathcal{A}$,
- a mass matrix M i.e. a linear map $M : \mathcal{H}_R \longrightarrow \mathcal{H}_L$,
- a certain number of coupling constants depending on the degree of reducibility of $\mathcal{H}_L \oplus \mathcal{H}_R$.

The data $(\mathcal{H}_L, \mathcal{H}_R, M)$ plays a fundamental role in non-commutative geometry where it generalizes the Dirac operator. It is called K-cycle.

On the output side we find a complete action of Yang-Mills-Higgs type with G the group of unitary elements in $\mathcal{A}$:

$$G = \{g \in \mathcal{A}, \quad gg^* = g^*g = 1\} \tag{9}$$

or possibly a subgroup thereof. In other words in Connes' approach the representation space $\mathcal{H}_S$ of scalars sits at the output end together with the representation space $\mathfrak{g}^{\mathbb{C}}$ of the gauge bosons. Likewise the quartic Higgs potential together with the Klein-Gordon Lagrangian for the scalars are found in the output accompanying the quartic Lagrangian for the gauge bosons.

In the following we first discuss Connes and Lott's rules for a finite dimensional (internal) space. Thereby we avoid the difficulties of functional analysis, only prerequisites being linear algebra, e.g. matrix multiplication. The second section deals with a particular infinite dimensional algebra, the functions on spacetime, and the Dirac operator. Model builders are well acquainted with this part of mathematics which also motivates Connes' rules. The third section is simply the tensor product of the first two. As we go along, the general rules are illustrated by an example. The simplest typical one we know strikingly resembles the Glashow-Salam-Weinberg model.

1. The internal space

An involution algebra $\mathcal{A}$ is an associative algebra with unit 1 and an involution *, i.e. an antilinear map from $\mathcal{A}$ into itself,

$$(a+b)^* = a^* + b^*, \qquad a, b \in \mathcal{A}, \tag{10}$$

$$(\lambda a)^* = \lambda^* a^*, \qquad \lambda \in \mathbb{R} \text{ or } \mathbb{C}, \tag{11}$$

with the properties:

$$a^{**} = a, \qquad 1^* = 1, \qquad (ab)^* = b^* a^*. \tag{12}$$

The two classical examples of involution algebra are:

(i) $M_n(\mathbb{C})$, the algebra of $n \times n$ matrices with complex entries. The involution is transposition and complex conjugation.

(ii) The algebra of functions from a manifold (spacetime) into the complex numbers. The involution is just complex conjugation.

The second example is infinite dimensional and commutative, while the first one is finite dimensional and non-commutative. Note that in this context the word non-abelian is non-fashionable.

A representation ρ of $\mathcal{A}$ over a Hilbert space $\mathcal{H}$ is a homomorphism from $\mathcal{A}$ into the operators on $\mathcal{H}$:

$$\rho : \mathcal{A} \longrightarrow \mathrm{End}(\mathcal{H}), \quad a \longmapsto \rho(a). \tag{13}$$

Here homomorphism refers to all given structures, addition, scalar multiplication, multiplication, unit and involution:

$$\rho(a+b) = \rho(a) + \rho(b), \tag{14}$$

$$\rho(\lambda a) = \lambda\rho(a), \tag{15}$$

$$\rho(ab) = \rho(a)\rho(b), \tag{16}$$

$$\rho(1) = 1, \tag{17}$$

$$\rho(a^*) = \rho(a)^*. \tag{18}$$

All representations will be supposed faithful, i.e. injective.

Model builders should note that the two choices, algebra and representation, are much more restricted than the choice of a group and a representation. Not every group G is the group of unitary elements in an algebra $\mathcal{A}$. The phenomenologically important $SU(3)$ is an example to which we shall have to come back. Every algebra representation yields a group representation of its group of unitaries, but most of these group representations cannot be obtained from an algebra representation. For example, the group $SU(2)$ has unitary representations of any dimension, 1, 2, 3... while the algebra of quaternions, whose group of unitaries is $SU(2)$, admits only one irreducible representation, the one of dimension 2. Indeed the tensor product of two algebra representations is not a representation because compatibility with the linear structure is lost. Also the popular singlet representation, "dark matter", is not available.

A K-cycle $(\mathcal{H}, \mathcal{D}, \chi)$ of an algebra $\mathcal{A}$ consists of a (faithful) representation ρ on a Hilbert space $\mathcal{H}$, of a self adjoint operator χ on $\mathcal{H}$, called chirality, satisfying

$$\chi^2 = 1 \tag{19}$$

and of a self adjoint operator $\mathcal{D}$ on $\mathcal{H}$, the generalized Dirac operator. Furthermore we suppose that $\rho(a)$ is even:

$$\rho(a)\chi = \chi\rho(a) \tag{20}$$

for all $a \in \mathcal{A}$ and that $\mathcal{D}$ is odd:

$$\mathcal{D}\chi = -\chi\mathcal{D}. \tag{21}$$

In the infinite dimensional case there will be additional conditions to be satisfied by $\mathcal{D}$.

In other words the representation ρ is reducible and decomposes into a left handed and a right handed part ρ_L and ρ_R living on the left handed and right handed Hilbert spaces

$$\mathcal{H}_L := \tfrac{1}{2}(1-\chi)\mathcal{H}, \quad \mathcal{H}_R := \tfrac{1}{2}(1+\chi)\mathcal{H}. \tag{22}$$

For a finite dimensional Hilbert space we can pick a basis such that

$$\chi = \begin{pmatrix} 1_L & 0 \\ 0 & -1_R \end{pmatrix}. \tag{23}$$

Then

$$\rho = \begin{pmatrix} \rho_L & 0 \\ 0 & \rho_R \end{pmatrix}, \qquad \mathcal{D} = \begin{pmatrix} 0 & M \\ M^* & 0 \end{pmatrix}, \tag{24}$$

with M a matrix of size $\dim \mathcal{H}_L \times \dim \mathcal{H}_R$, the mass matrix.

Example. Let $\mathcal{A} := M_2(\mathbb{C}) \oplus \mathbb{C}$ and denote its elements by (a,b), a being a 2×2 matrix and b a complex number. We define a K-cycle by

$$\mathcal{H}_L := \mathbb{C}^2, \qquad \mathcal{H}_R := \mathbb{C}, \tag{25}$$

$$\rho_L(a,b) := a, \qquad \rho_R(a,b) := b, \tag{26}$$

with

$$M := \begin{pmatrix} m \\ 0 \end{pmatrix}, \quad m \in \mathbb{C}. \tag{27}$$

Given an (involution) algebra and a K-cycle we now want to construct a differential algebra. Let us recall the axioms of a differential algebra Ω. It is a graded vector space

$$\Omega = \bigoplus_{p\in\mathbb{N}_0} \Omega^p. \tag{28}$$

We denote its associative product by juxtaposition:

$$\Omega^p \times \Omega^q \longrightarrow \Omega^{p+q}, \quad (\phi,\psi) \longmapsto \phi\psi. \tag{29}$$

Furthermore Ω is equipped with a differential δ that is a linear map

$$\delta : \Omega^p \longrightarrow \Omega^{p+1}, \quad \phi \longmapsto \delta\phi \tag{30}$$

with two properties. It is nilpotent,

$$\delta^2 = 0 \tag{31}$$

and obeys a graded Leibniz rule,

$$\delta(\phi\psi) = (\delta\phi)\psi + (-1)^p \phi\delta\psi, \quad \phi \in \Omega^p. \tag{32}$$

The differential algebra is called (graded) commutative if in addition its product satisfies

$$\phi\psi = (-1)^{pq}\psi\phi, \quad \phi \in \Omega^p, \ \psi \in \Omega^q. \tag{33}$$

From our given algebra $\mathcal{A}$ we now construct first an auxiliary differential algebra $\hat{\Omega}\mathcal{A}$, the so called universal differential envelop of $\mathcal{A}$:

$$\hat{\Omega}^0\mathcal{A} := \mathcal{A}, \tag{34}$$

$\hat{\Omega}^1\mathcal{A}$ is generated by symbols $\delta a, a \in \mathcal{A}$ with relations

$$\delta 1 = 0, \qquad \delta(ab) = (\delta a)b + a\delta b. \tag{35}$$

Therefore $\hat{\Omega}^1\mathcal{A}$ consists of finite sums of terms of the form $a_0\delta a_1$,

$$\hat{\Omega}^1\mathcal{A} = \left\{ \sum_j a_0^j \delta a_1^j, \quad a_0^j, a_1^j \in \mathcal{A} \right\} \tag{36}$$

and likewise for higher p

$$\hat{\Omega}^p\mathcal{A} = \left\{ \sum_j a_0^j \delta a_1^j ... \delta a_p^j, \quad a_q^j \in \mathcal{A} \right\}. \tag{37}$$

The differential δ is defined by

$$\delta(a_0\delta a_1 ... \delta a_p) := \delta a_0 \delta a_1 ... \delta a_p. \tag{38}$$

Two remarks: The universal differential envelope $\hat{\Omega}\mathcal{A}$ of a commutative algebra $\mathcal{A}$ is not necessarily graded commutative. The universal differential envelope of any algebra has no cohomology. This means that every closed form $\hat{\phi}$ of degree $p \geq 1, \delta\hat{\phi} = 0$, is exact, $\hat{\phi} = \delta\hat{\psi}$ for some $(p-1)$-form $\hat{\psi}$.

The involution * can be extended from the algebra $\mathcal{A}$ to its universal differential envelope $\hat{\Omega}^1\mathcal{A}$ by putting

$$(\delta a)^* := \delta(a^*) =: \delta a^* \tag{39}$$

and of course

$$(\phi\psi)^* = \psi^*\phi^*. \tag{40}$$

Note that Connes defines $(\delta a)^* := -\delta(a^*)$, which amounts to replacing δ by $i\delta$.

Our next step is to extend the representation ρ from the algebra $\mathcal{A}$ to its universal differential envelope $\hat{\Omega}\mathcal{A}$. This extension is the central piece of Connes' algorithm and deserves a new name:

$$\pi : \hat{\Omega}\mathcal{A} \longrightarrow \mathrm{End}(\mathcal{H}), \quad \hat{\phi} \longmapsto \pi(\hat{\phi}), \tag{41}$$
$$\pi(a_0\delta a_1 ... \delta a_p) := (-i)^p \rho(a_0)[\mathcal{D}, \rho(a_1)] ... [\mathcal{D}, \rho(a_p)].$$

Note that in Connes' notations there is no factor $(-i)^p$ on the r.h.s. A straightforward calculation shows that π is in fact a representation of $\hat{\Omega}\mathcal{A}$ as involution algebra, and we are tempted to define also a differential, again denoted by δ, on $\pi(\hat{\Omega}\mathcal{A})$ by

$$\delta\pi(\hat{\phi}) := \pi(\delta\hat{\phi}). \tag{42}$$

However, this definition does not make sense if there are forms $\hat{\phi}$ with $\pi(\hat{\phi}) = 0$ and $\pi(\delta\hat{\phi}) \neq 0$. By dividing out these unpleasant forms we shall construct a new differential algebra $\Omega\mathcal{A}$, the real thing. It will of course depend on the chosen K-cycle.

The kernel of π is a (bilateral) ideal in the differential algebra $\hat{\Omega}\mathcal{A}$. We turn it into a differential ideal J (J for junk):

$$J := \ker\pi + \delta\ker\pi =: \bigoplus_p J^p \tag{43}$$

with

$$J^p = (\ker\pi)^p + \delta(\ker\pi)^{p-1} \tag{44}$$

and divide it out:

$$\Omega\mathcal{A} := \hat{\Omega}\mathcal{A}/J. \tag{45}$$

On the quotient now, the differential (42) is well defined. Degree by degree we have:

$$\Omega^0\mathcal{A} = \hat{\Omega}^0\mathcal{A} \cong \rho(\mathcal{A}) \tag{46}$$

because ρ is faithful and $J^0 = (\ker\pi)^0 = 0$,

$$\Omega^1\mathcal{A} = \hat{\Omega}^1\mathcal{A}/(\ker\pi)^1 \cong \pi(\hat{\Omega}^1\mathcal{A}) \tag{47}$$

because $J^1 = (\ker\pi)^1$, and in degree $p \geq 2$

$$\Omega^p\mathcal{A} \cong \pi(\hat{\Omega}^p\mathcal{A})/\pi(\delta(\ker\pi)^{p-1}). \tag{48}$$

While $\hat{\Omega}\mathcal{A}$ has no cohomology $\Omega\mathcal{A}$ does in general. In fact let us anticipate, if $\mathcal{F}$ is the algebra of complex functions on a compact spin manifold M of even dimensions and if the K-cycle is obtained from the Dirac operator then $\Omega\mathcal{F}$ is de Rham's differential algebra of differential forms on M.

We come back to our finite dimensional case. Remember that the elements of the auxiliary differential algebra $\hat{\Omega}\mathcal{A}$ that we introduced for book keeping purposes only are abstract entities defined in terms of symbols and relations. On the other hand, with the above isomorphisms, the elements of $\Omega\mathcal{A}$, the "forms", are operators on the Hilbert space $\mathcal{H}$ of the K-cycle, i.e. concrete matrices of complex numbers.

Examples. Before continuing our example above let us mention a class of trivial examples that deserve the name *vector like models*. The algebra $\mathcal{A}$ is arbitrary, left and right representations are equal $\rho_L = \rho_R$, and the mass matrix appearing in the "Dirac" operator $\mathcal{D}$ is a multiple of the identity matrix $M = \lambda 1$, $\lambda \in \mathbb{C}$. We shall see that these models will produce Yang-Mills theories with unbroken parity and unbroken gauge symmetry as electromagnetism and chromodynamics. Any vector like model has trivial differential algebra,

$$\Omega^0\mathcal{A}=\mathcal{A}, \qquad \Omega^p\mathcal{A}=0, \quad p=1,2,3,\ldots \tag{49}$$

Coming back to our serious example recall:

$$\rho(a,b)=\begin{pmatrix} a & 0\\ 0 & b\end{pmatrix}, \qquad \mathcal{D}=\begin{pmatrix} 0 & M\\ M^* & 0\end{pmatrix}. \tag{50}$$

We need the commutator

$$\begin{aligned}[\mathcal{D},\rho(a,b)] &= \begin{pmatrix} 0 & M\rho_R(b)-\rho_L(a)M\\ M^*\rho_L(a)-\rho_R(b)M^* & 0\end{pmatrix}\\ &= \begin{pmatrix} 0 & Mb-aM\\ M^*a-bM^* & 0\end{pmatrix}\\ &= \begin{pmatrix} 0 & -(a-B)M\\ M^*(a-B) & 0\end{pmatrix},\end{aligned} \tag{51}$$

where we have put $B := b1$. A general element in $\Omega^1\mathcal{A}$ is of the form

$$\pi((a_0,b_0)\delta(a_1,b_1))=-i\begin{pmatrix} 0 & -a_0(a_1-B_1)M\\ M^*B_0(a_1-B_1) & 0\end{pmatrix} \tag{52}$$

and as vector space

$$\Omega^1\mathcal{A}=\left\{i\begin{pmatrix} 0 & hM\\ M^*\tilde{h}^* & 0\end{pmatrix},\ h,\tilde{h}\in M_2(\mathbb{C})\right\}. \tag{53}$$

Likewise a general element in $\pi(\hat{\Omega}^2\mathcal{A})$ is

$$\begin{aligned}&\pi((a_0,b_0)\delta(a_1,b_1)\delta(a_2,b_2))\\ &=\begin{pmatrix} a_0(a_1-B_1)MM^*(a_2-B_2) & 0\\ 0 & M^*B_0(a_1-B_1)(a_2-B_2)M\end{pmatrix}\\ &=\begin{pmatrix} \Delta a_0(a_1-B_1)(a_2-B_2) \\ \quad +\Delta a_0(a_1-B_1)\sigma_3(a_2-B_2) & 0\\ 0 & M^*B_0(a_1-B_1)(a_2-B_2)M\end{pmatrix}\end{aligned} \tag{54}$$

where we have used the decomposition

$$MM^*=\begin{pmatrix} |m|^2 & 0\\ 0 & 0\end{pmatrix}=\Delta 1+\Delta\sigma_3 \tag{55}$$

with

$$\Delta := \tfrac{1}{2}|m|^2, \qquad \sigma_3 := \begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix}. \tag{56}$$

A general element in $(\ker\pi)^1$ is a finite sum of the form

$$\sum_j (a_0^j, b_0^j)\delta(a_1^j, b_1^j) \tag{57}$$

with the two conditions

$$\left[\sum_j a_0^j(a_1^j - B_1^j)\right] M = 0, \tag{58}$$

$$M^* \left[\sum_j B_0^j(a_1^j - B_1^j)\right] = 0. \tag{59}$$

Therefore the corresponding general element in $\pi(\delta(\ker\pi)^1)$ is

$$\begin{pmatrix} \Delta\sum_j(a_0^j - B_0^j)(a_1^j - B_1^j) + \Delta\sum_j(a_0^j - B_0^j)\sigma_3(a_1^j - B_1^j) & 0 \\ 0 & 0 \end{pmatrix} \tag{60}$$

still subject to the two conditions. We have the following inclusion

$$\begin{aligned} \pi(\delta(\ker\pi)^1) &\supset \left\{ \begin{pmatrix} \Delta\sum_j a_0^j\sigma_3 a_1^j & 0 \\ 0 & 0 \end{pmatrix}, \quad \sum_j a_0^j a_1^j = 0 \right\} \\ &= \left\{ \begin{pmatrix} \Delta k & 0 \\ 0 & 0 \end{pmatrix}, \quad k \in M_2(\mathbb{C}) \right\}. \end{aligned} \tag{61}$$

To prove the last equality we note that the subspace is a bilateral ideal in the rhs. Furthermore the subspace contains the non-zero element with:

$$a_0 := \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}, \qquad a_1 := \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \tag{62}$$

$$a_0 a_1 = 0, \qquad a_0\sigma_3 a_1 = \begin{pmatrix} 0 & 0 \\ 2 & 2 \end{pmatrix}. \tag{63}$$

The algebra $M_2(\mathbb{C})$ being simple the subspace is the entire algebra. Consequently

$$\pi(\delta(\ker\pi)^1) = \left\{ \begin{pmatrix} \Delta k & 0 \\ 0 & 0 \end{pmatrix}, \quad k \in M_2(\mathbb{C}) \right\}. \tag{64}$$

Now we have to compute the quotient

$$\Omega^2\mathcal{A} = \pi(\hat{\Omega}^2\mathcal{A})/\pi(\delta(\ker\pi)^1). \tag{65}$$

We are tempted to conclude

$$\Omega^2\mathcal{A} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & M^*cM \end{pmatrix}, \quad c \in M_2(\mathbb{C}) \right\}. \tag{66}$$

The problem with this conclusion is that a quotient of vector spaces consists of classes and is not canonically a subspace. The situation is simpler if our vector space comes equipped with a scalar product, in which case there is a privileged representative in each class, the one orthogonal to the subspace $\pi(\delta(\ker\pi)^1)$. This canonical choice allows us to consider the quotient as subspace.

Since the elements of $\pi(\hat{\Omega}\mathcal{A})$ are operators on the Hilbert space $\mathcal{H}$, i.e. concrete matrices, they have a natural scalar product defined by

$$\langle\hat{\phi},\hat{\psi}\rangle := \mathrm{tr}(\hat{\phi}^*\hat{\psi}), \quad \hat{\phi},\hat{\psi} \in \pi(\hat{\Omega}^p\mathcal{A}) \tag{67}$$

for forms of equal degree and zero for the scalar product of two forms of different degree. With this scalar product $\Omega\mathcal{A}$ is a subspace of $\pi(\hat{\Omega}\mathcal{A})$ and by definition orthogonal to $\pi(J) = \pi(\delta\ker\pi)$. Now our conclusion in the example makes sense. As a subspace $\Omega\mathcal{A}$ inherits a scalar product which deserves a special name $(\ ,\ )$. It is given by

$$(\phi,\psi) = \mathrm{tr}(\phi^* P\psi), \quad \phi,\psi \in \Omega^p\mathcal{A} \tag{68}$$

where P is the orthogonal projector in $\pi(\hat{\Omega}\mathcal{A})$ onto the ortho-complement of J and ϕ and ψ are any representatives in their classes. Again the scalar product vanishes for forms with different degree. For real algebras the scalar products are defined as the real part of the trace.

Let us recapitulate our example:

$$\Omega^0\mathcal{A} = \left\{\begin{pmatrix} a & 0 \\ 0 & b\end{pmatrix}, \quad a \in M_2(\mathbb{C}),\ b \in \mathbb{C}\right\}, \tag{69}$$

$$\Omega^1\mathcal{A} = \left\{ i\begin{pmatrix} 0 & hM \\ M^*\tilde{h}^* & 0\end{pmatrix}, \ h,\tilde{h} \in M_2(\mathbb{C})\right\}, \tag{70}$$

$$\Omega^2\mathcal{A} = \left\{\begin{pmatrix} 0 & 0 \\ 0 & M^*cM\end{pmatrix}, \quad c \in M_2(\mathbb{C})\right\}. \tag{71}$$

Since π is a homomorphism of involution algebras the product in $\Omega\mathcal{A}$ is given by matrix multiplication followed by the projection

$$P = \begin{pmatrix} 0 & 0 \\ 0 & 1\end{pmatrix} \tag{72}$$

and the involution is given by transposition complex conjugation. It is in order to calculate the differential δ that we need the complicated construction above:

$$\begin{aligned} &\delta : \Omega^0\mathcal{A} \longrightarrow \Omega^1\mathcal{A} \\ &\begin{pmatrix} a & 0 \\ 0 & b\end{pmatrix} \longmapsto i\begin{pmatrix} 0 & (a-B)M \\ -M^*(a-B) & 0\end{pmatrix}, \end{aligned} \tag{73}$$

$$\begin{aligned} &\delta : \Omega^1\mathcal{A} \longrightarrow \Omega^2\mathcal{A} \\ i&\begin{pmatrix} 0 & hM \\ M^*\tilde{h}^* & 0\end{pmatrix} \longmapsto \begin{pmatrix} 0 & 0 \\ 0 & M^*(h+\tilde{h}^*)M\end{pmatrix}. \end{aligned} \tag{74}$$

Let us note that not every example can be calculated as explicitly as the above one. Just add a non-diagonal entry in the mass matrix M of our example and δ will not have the simple form indicated.

We are now ready to make a first contact with gauge theories. Consider the vector space of antihermitian 1-forms

$$\left\{H \in \Omega^1\mathcal{A}, H^* = -H\right\}. \tag{75}$$

A general element H is of the form

$$H = i\begin{pmatrix} 0 & hM \\ M^*h^* & 0 \end{pmatrix}, \quad h \in M_2(\mathbb{C}). \tag{76}$$

These elements are called Higgses or gauge potentials. In fact the space of gauge potentials carries an affine representation of the group of unitaries

$$G = \{g \in \mathcal{A}, gg^* = g^*g = 1\} = U(2) \times U(1) \tag{77}$$

defined by

$$\begin{aligned} H^g &:= \rho(g)H\rho(g^{-1}) + \rho(g)\delta\rho(g^{-1}) \\ &= \rho(g)H\rho(g^{-1}) + (-i)\rho(g)[\mathcal{D}, \rho(g^{-1})] \\ &=: i\begin{pmatrix} 0 & h^gM \\ M^*(h^g)^* & 0 \end{pmatrix} \end{aligned} \tag{78}$$

with

$$h^g = \rho_L(g)[h-1]\rho_R(g^{-1}) + 1. \tag{79}$$

H^g is the "gauge transformed of H". As usual every gauge potential H defines a covariant derivative $\delta + H$, covariant under the left action of G on $\Omega\mathcal{A}$:

$${}^g\psi := \rho(g)\psi, \quad \psi \in \Omega\mathcal{A} \tag{80}$$

which means

$$(\delta + H^g)\ {}^g\psi = {}^g\left[(\delta + H)\psi\right]. \tag{81}$$

As usual we define the curvature C of H by

$$C := \delta H + H^2 \ \in \Omega^2\mathcal{A}. \tag{82}$$

Note that here and later H^2 is considered as element of $\Omega^2\mathcal{A}$, which means it is the projection P applied to $H^2 \in \pi(\hat{\Omega}^2\mathcal{A})$. The curvature C is a hermitian 2-form with *homogeneous* gauge transformations

$$C^g := \delta(H^g) + (H^g)^2 = \rho(g)C\rho(g^{-1}). \tag{83}$$

Finally we define the "Higgs potential" $V(H)$, a functional on the space of gauge potentials, by

$$V(H) := (C, C) = \mathrm{tr}[(\delta H + H^2)P(\delta H + H^2)]. \tag{84}$$

It is a polynomial of degree 4 in H with real, non-negative values. Furthermore it is gauge invariant,

$$V(H^g) = V(H) \tag{85}$$

because of the homogeneous transformation property of the curvature C and because the orthogonal projector P commutes with all gauge transformations

$$\rho(g)P = P\rho(g). \tag{86}$$

The transformation law for H motivates the following change of variables

$$\Phi := i\begin{pmatrix} 0 & \phi M \\ M^*\phi^* & 0 \end{pmatrix} := H - i\mathcal{D}. \tag{87}$$

In other words

$$\phi = h - 1. \tag{88}$$

The new variable Φ transforms homogeneously

$$\Phi^g = \rho(g)\Phi\rho(g^{-1}) \tag{89}$$

or

$$\phi^g = \rho_L(g)\phi\rho_R(g^{-1}) \tag{90}$$

where the differential is of course considered gauge invariant

$$\mathcal{D}^g = \mathcal{D}. \tag{91}$$

In our example the Higgs h is a complex 2×2 matrix, which in terms of ϕ decomposes under gauge transformations into two complex doublets, the two column vectors ϕ_1 and ϕ_2 of ϕ. The second doublet ϕ_2 disappears due to the special form of the mass matrix. The curvature is readily calculated

$$C := \delta H + H^2 = \begin{pmatrix} 0 & 0 \\ 0 & M^*cM \end{pmatrix} \tag{92}$$

with

$$c = h + h^* - h^*h = 1 - \phi^*\phi. \tag{93}$$

The Higgs potential is

$$\begin{aligned} V(H) &= \mathrm{tr}\left[C^2\right] = \ \mathrm{tr}\left[(M^*(1-\phi^*\phi)M\right]^2 \\ &= |m|^4(1-\phi_1^*\phi_1)^2. \end{aligned} \tag{94}$$

Its most interesting feature is that it breaks the gauge symmetry spontaneously. Indeed the only gauge invariant point in the space of gauge potentials or Higgses is $\phi = 0$. This point is not a minimum of the Higgs potential.

2. Spacetime

In this section our algebra is infinite dimensional, the algebra of differentiable, complex valued functions on spacetime M

$$\mathcal{F} := \mathcal{C}^\infty(M). \tag{95}$$

The K-cycle is defined by the Dirac operator. We sketch how the differential algebra $\Omega\mathcal{F}$ reproduces the ordinary differential forms on M. For simplicity spacetime is taken flat, compact and Euclidean. To define the Dirac operator we also need a spin structure. We denote by $\mathcal{S}$ the Hilbert space of the K-cycle. $\mathcal{S}$ consists of square integrable spinors

$$\Psi = \begin{pmatrix} \Psi_1(x) \\ \Psi_2(x) \\ \Psi_3(x) \\ \Psi_4(x) \end{pmatrix}. \tag{96}$$

The representation of $\mathcal{F}$ on $\mathcal{S}$ is simply by multiplication and is denoted by $\underline{\cdot}$,

$$(\underline{f}\Psi)(x) := f(x)\Psi(x), \quad f \in \mathcal{F},\ \Psi \in \mathcal{S}. \tag{97}$$

The odd operator of the K-cycle is the (true) Dirac operator

$$\not\partial\Psi := i\sum_{\mu=0}^{3} \gamma^\mu \frac{\partial}{\partial x^\mu}\Psi. \tag{98}$$

Our gamma matrices are self adjoint,

$$\gamma^0 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \gamma^1 := \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \tag{99}$$

$$\gamma^2 := \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \gamma^3 := \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}. \tag{100}$$

They satisfy the anticommutation relation

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}1 \tag{101}$$

with the flat Euclidean metric

$$\eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{102}$$

The chirality operator is by definition

$$\gamma_5 := \gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \tag{103}$$

It is self adjoint, its square is one as postulated and since it anticommutes with all other gamma matrices

$$\gamma^\mu\gamma_5 + \gamma_5\gamma^\mu = 0, \tag{104}$$

the Dirac operator is odd

$$\partial\!\!\!/\gamma_5 + \gamma_5\partial\!\!\!/ = 0. \tag{105}$$

The Dirac operator $\partial\!\!\!/$ has additional algebraic properties:
- The commutator $[\partial\!\!\!/, \underline{f}]$ is a bounded operator for all $f \in \mathcal{F}$.
- The spectrum of $\partial\!\!\!/$ is discrete and the eigenvalues λ_n of $|\partial\!\!\!/|$ arranged in an increasing sequence are of order $n^{(1/d)}$ for d dimensional manifolds M, d even, for us $d = 4$.

These two properties are added to the axioms of an abstract K-cycle. They will be needed later to define a trace.

We denote by d the differential in the universal differential envelope $\hat{\Omega}\mathcal{F}$ and by π_D, D for Dirac, the algebra homomorphism

$$\pi_D : \hat{\Omega}\mathcal{F} \longrightarrow \mathrm{End}(\mathcal{S}), \quad \hat{\phi} \longmapsto \pi_D(\hat{\phi}), \tag{106}$$

$$\pi_D(f_0\mathrm{d}f_1...\mathrm{d}f_p) := (-i)^p \underline{f}_0[\partial\!\!\!/, \underline{f}_1]...[\partial\!\!\!/, \underline{f}_p]. \tag{107}$$

We need the commutator

$$\begin{aligned} [\partial\!\!\!/, \underline{f}]\Psi &= i\gamma^\mu \frac{\partial}{\partial x^\mu}(f\Psi) - if\gamma^\mu\frac{\partial}{\partial x^\mu}\Psi \\ &= i\left[\gamma^\mu\frac{\partial}{\partial x^\mu}f\right]\Psi. \end{aligned} \tag{108}$$

(We use Einstein's summation convention.) Therefore

$$[\partial\!\!\!/, \underline{f}] = i\gamma^\mu\frac{\partial}{\partial x^\mu}f =: i\gamma(\mathrm{d}f) \tag{109}$$

with

$$\mathrm{d}f = \left[\frac{\partial}{\partial x^\mu}f\right]\mathrm{d}x^\mu. \tag{110}$$

At this point already we see that the restriction to flat spacetime can be dropped. The Dirac operator on curved manifolds,

$$i\gamma^\mu(x)\left[\frac{\partial}{\partial x^\mu} + \omega_\mu\right], \tag{111}$$

differs from the flat one in two respects: the gamma matrices are x dependent, no problem in the above commutator, and an additional algebraic term, a spin connection $\omega = \omega_\mu \mathrm{d}x^\mu$ valued in $so(4)$ appears but drops out from the commutator. Since the Dirac operator only shows up in commutators Connes' algorithm works on any Riemannian manifold.

The representation of functions by multiplication on spinors is faithful, of course, and

$$\Omega^1\mathcal{F} \cong \pi_D(\hat{\Omega}^1\mathcal{F})\,. \tag{112}$$

A general element of the rhs is a finite sum of terms

$$\pi_D(f_0 \mathrm{d} f_1), \quad f_0, f_1 \in \mathcal{F}. \tag{113}$$

It is identified with the differential 1-form on M

$$f_0 \mathrm{d} f_1 \quad \in \Omega^1 M. \tag{114}$$

For 2-forms the situation is less trivial, we must compute the junk $\pi_D(\mathrm{d}(\ker\pi_D)^1)$. Consider

$$h^{-1}\mathrm{d}h + h\mathrm{d}h^{-1}\,, \tag{115}$$

an element in $\hat{\Omega}^1\mathcal{F}$ where $h \in \mathcal{F}$ is a non-vanishing function, $h^{-1}(x) = 1/h(x)$. As $\hat{\Omega}\mathcal{F}$ is not graded commutative this element does not vanish!

$$h^{-1}\mathrm{d}h + h\mathrm{d}h^{-1} \neq h^{-1}\mathrm{d}h + (\mathrm{d}h^{-1})h = \mathrm{d}(h^{-1}h) = \mathrm{d}1 = 0. \tag{116}$$

Its image under π_D, however, does vanish:

$$\begin{aligned}\pi_D(h^{-1}\mathrm{d}h + h\mathrm{d}h^{-1}) &= \gamma(h^{-1}\mathrm{d}h + h\mathrm{d}h^{-1})\\ &= \gamma(h^{-1}\mathrm{d}h + (\mathrm{d}h^{-1})h) = 0.\end{aligned} \tag{117}$$

Therefore the considered element is in $(\ker\pi_D)^1$ and the corresponding element in $\pi_D(\mathrm{d}(\ker\pi_D)^1)$ is

$$\begin{aligned}\pi_D(\mathrm{d}h^{-1}\mathrm{d}h + \mathrm{d}h\mathrm{d}h^{-1}) &= \gamma(\mathrm{d}h^{-1})\gamma(\mathrm{d}h) + \gamma(\mathrm{d}h)\gamma(\mathrm{d}h^{-1})\\ &= \gamma^\mu\left(\frac{\partial}{\partial x^\mu}h^{-1}\right)\gamma^\nu\frac{\partial}{\partial x^\nu}h + \gamma^\nu\left(\frac{\partial}{\partial x^\nu}h\right)\gamma^\mu\frac{\partial}{\partial x^\mu}h^{-1}\\ &= \left[\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu\right]\left(\frac{\partial}{\partial x^\mu}h^{-1}\right)\frac{\partial}{\partial x^\nu}h\\ &= -\left(\frac{2}{h^2}\eta^{\mu\nu}\left(\frac{\partial}{\partial x^\mu}h\right)\frac{\partial}{\partial x^\nu}h\right)1.\end{aligned} \tag{118}$$

By linear combination

$$\pi_D(\mathrm{d}(\ker\pi_D)^1) = \{f1, \quad f \in \mathcal{F}\}\,. \tag{119}$$

On the other hand

$$\pi_D(\hat{\Omega}^2\mathcal{F}) = \{f_{\mu\nu}\gamma^\mu\gamma^\nu,\quad f_{\mu\nu}\in\mathcal{F}\}\,, \tag{120}$$

$$\pi_D(\mathrm{d}f_1\mathrm{d}f_2+\mathrm{d}f_2\mathrm{d}f_1) = \left(2\eta^{\mu\nu}\frac{\partial}{\partial x^\mu}f_1\frac{\partial}{\partial x^\nu}f_2\right)1\,. \tag{121}$$

After passage to the quotient $\pi_D(\mathrm{d}f_1)$ and $\pi_D(\mathrm{d}f_2)$ anticommute whereas they did not anticommute in $\pi_D(\hat{\Omega}^2\mathcal{F})$ and we may now identify a general element

$$\pi_D(f_0\mathrm{d}f_1\mathrm{d}f_2)\quad\in\Omega^2\mathcal{F} \tag{122}$$

with the differential 2-form on M

$$f_0\mathrm{d}f_1\mathrm{d}f_2\quad\in\Omega^2M. \tag{123}$$

As in Section 1 we have treated the quotient space like a subspace, which is legitimate only in presence of an appropriate scalar product. Again this scalar product will be defined in terms of the involution and a trace.

The involution that ΩM inherits from $\Omega\mathcal{F}$ via the sketched isomorphism is with our conventions

$$(f_0\mathrm{d}f_1\mathrm{d}f_2...\mathrm{d}f_p)^* = (-1)^{(1/2)p(p-1)}\bar{f}_0\mathrm{d}\bar{f}_1\mathrm{d}\bar{f}_2...\mathrm{d}\bar{f}_p. \tag{124}$$

The definition of a trace is delicate because now our Hilbert space $\mathcal{S}$ is infinite dimensional. For any bounded, positive operator Q on $\mathcal{S}$ we define the *Dixmier trace* tr_ω by

$$\mathrm{tr}_\omega(Q|\partial\!\!\!/|^{-d}) := \lim_{N\to\infty}\frac{1}{\log N}\sum_{n=1}^{N}\lambda_n\,, \tag{125}$$

where $d=\dim M=4$ and the λ_n are the eigenvalues of $Q|\partial\!\!\!/|^{-d}$ arranged in a decreasing sequence discarding the zero modes of the Dirac operator. Now we proceed as in the finite dimensional case $(d=0)$ and define a scalar product on $\pi_D(\hat{\Omega}\mathcal{F})$ by

$$\langle\hat{\phi}_D,\hat{\psi}_D\rangle := \mathrm{tr}_\omega(\hat{\phi}_D^*\hat{\psi}_D|\partial\!\!\!/|^{-4}),\quad \hat{\phi}_D,\hat{\psi}_D\in\pi_D(\hat{\Omega}\mathcal{F}). \tag{126}$$

Note that $\hat{\phi}_D$ is bounded because $[\partial\!\!\!/,\underline{f}]$ is. This scalar product can be computed to be

$$\langle\hat{\phi}_D,\hat{\psi}_D\rangle = \frac{1}{32\pi^2}\int_M \mathrm{tr}_4\left[\hat{\phi}_D^*\hat{\psi}_D\right]\mathrm{d}^4x \tag{127}$$

independently of the four dimensional manifold M. tr_4 denotes the trace over the gamma matrices. With this scalar product $\Omega\mathcal{F}$ is a subspace of $\pi_D(\hat{\Omega}\mathcal{F})$ and by definition orthogonal to $J=\ker\pi_D+\mathrm{d}\ker\pi_D$. As subspace $\Omega\mathcal{F}$ inherits a scalar product $(\cdot,\cdot)$ given by

$$(\phi_D,\psi_D) = \langle\phi_D,P_D\psi_D\rangle,\quad \phi_D,\psi_D\in\Omega^p\mathcal{F}\,, \tag{128}$$

where P_D is the orthogonal projector in $\pi_D(\hat{\Omega}\mathcal{F})$ onto the ortho-complement of $\pi(J)$ and ϕ_D and ψ_D are any representatives in their classes. Thanks to well known results for

$\mathrm{tr}_4\left[\gamma^{\mu_1}v_{\mu_1}...\gamma^{\mu_q}v_{\mu_q}\right]$ this scalar product now vanishes for forms with different degree. By the above isomorphism between $\Omega\mathcal{F}$ and ΩM the differential forms inherit a scalar product still denoted by $(\cdot,\cdot)$

$$(\phi_D,\psi_D)=\frac{1}{8\pi^2}\int_M \phi_D^* * \psi_D,\quad \phi_D,\psi_D\in\Omega^p M, \tag{129}$$

where the Hodge star $*\cdot$ should not be confused with the involution $\cdot^*$.

As an example let us consider the flat 4-torus, $M=T^4$ with all four circumferences measuring 2π and let us compute the eigenvalues of the Dirac operator

$$\partial\!\!\!/\Psi=\begin{pmatrix} i\frac{\partial}{\partial x^0} & 0 & -\frac{\partial}{\partial x^3} & -\frac{\partial}{\partial x^1}+i\frac{\partial}{\partial x^2}\\ 0 & i\frac{\partial}{\partial x^0} & -\frac{\partial}{\partial x^1}-i\frac{\partial}{\partial x^2} & \frac{\partial}{\partial x^3}\\ \frac{\partial}{\partial x^3} & \frac{\partial}{\partial x^1}-i\frac{\partial}{\partial x^2} & -i\frac{\partial}{\partial x^0} & 0\\ \frac{\partial}{\partial x^1}+i\frac{\partial}{\partial x^2} & -\frac{\partial}{\partial x^3} & 0 & -i\frac{\partial}{\partial x^0}\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\\ \Psi_3\\ \Psi_4\end{pmatrix}=\lambda\begin{pmatrix}\Psi_1\\ \Psi_2\\ \Psi_3\\ \Psi_4\end{pmatrix}. \tag{130}$$

After a Fourier transform

$$\Psi_A =: \sum_{j_0,\dots,j_3\in\mathbb{Z}} c_A(j_0,\dots,j_3)\exp(-ij_\mu x^\mu),\quad A=1,2,3,4 \tag{131}$$

this equation reads

$$\begin{pmatrix} j_0 & 0 & ij_3 & ij_1+j_2\\ 0 & j_0 & ij_1-j_2 & -ij_3\\ -ij_3 & -ij_1-j_2 & -j_0 & 0\\ -ij_1+j_2 & ij_3 & 0 & -j_0\end{pmatrix}\begin{pmatrix}c_1\\ c_2\\ c_3\\ c_4\end{pmatrix}=\lambda\begin{pmatrix}c_1\\ c_2\\ c_3\\ c_4\end{pmatrix}. \tag{132}$$

Its characteristic equation is

$$\left[\lambda^2-(j_0^2+j_1^2+j_2^2+j_3^2)^2\right]^2 = 0 \tag{133}$$

and for fixed j_μ each eigenvalue

$$\lambda = \pm\sqrt{j_0^2+j_1^2+j_2^2+j_3^2} \tag{134}$$

has multiplicity two. Therefore asymptotically for large $|\lambda|$ there are $4|\lambda|^4B_4$ eigenvalues (counted with their multiplicity) whose absolute values are smaller than $|\lambda|$. $B_4=\pi^2/2$ denotes the volume of the unit ball in $\mathbb{R}^4$. Let us arrange the absolute values of the eigenvalues in an increasing sequence. Taking due account of their multiplicities we have for large n

$$|\lambda_n|\approx\left(n/2\pi^2\right)^{1/4} \tag{135}$$

and we can check the Dixmier trace in Eq. (127) for instance with $\hat\phi_D=\hat\psi_D=1$,

$$\langle 1,1\rangle=\mathrm{tr}_\omega(|\partial\!\!\!/|^{-4})=\lim_{N\to\infty}\frac{1}{\log N}\sum_{n=1}^{N}|\lambda_n|^{-4}$$

$$= \lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{2\pi^2}{n} = \lim_{N\to\infty} \frac{1}{\log N} \int_1^N \frac{2\pi^2}{n}\,\mathrm{d}n$$

$$= 2\pi^2 = \frac{1}{32\pi^2} \int_M \mathrm{tr}_4[1]\mathrm{d}^4x. \tag{136}$$

Let us come back to the general case. The group of unitaries is now infinite dimensional

$$G = \{g \in \mathcal{F},\ gg^* = g^*g = 1\} = \{M \to U(1)\}, \tag{137}$$

the gauge group $U(1)$. A gauge potential is simply a differential 1-form A with values in the Lie algebra $u(1)$, the vector potential of electromagnetism. Its curvature F is the $u(1)$ valued 2-form

$$F := \mathrm{d}A + A^2 = \mathrm{d}A \tag{138}$$

(the differential algebra $\Omega\mathcal{F}$ being graded commutative $A^2 \equiv 0$) and the Higgs potential is precisely the Maxwell action

$$(F, F) = \int_M F * F. \tag{139}$$

3. The tensor product

Remember the description of spinning particles in quantum mechanics. Particles with spin s come in $2s+1$ dimensional unitary representations of the group $SU(2)$. Position in space enters the picture via the tensor product of this finite dimensional Hilbert space with the space of square integrable functions. In this spirit we shall now turn the Higgses into genuine Higgs fields by tensorizing $\mathcal{A}$ and $\mathcal{F}$.

Let us denote this tensor product by

$$\mathcal{A}_t := \mathcal{F} \otimes \mathcal{A}. \tag{140}$$

The algebra $\mathcal{A}_t$ admits a natural K-cycle $(\mathcal{H}_t, \mathcal{D}_t, \chi_t)$, the tensor product of the K-cycles $(\mathcal{S}, \not{\partial}, \gamma_5)$ on $\mathcal{F}$ and $(\mathcal{H}, \mathcal{D}, \chi)$ on $\mathcal{A}$. The Hilbert space

$$\mathcal{H}_t := \mathcal{S} \otimes \mathcal{H} \tag{141}$$

carries the representation

$$\rho_t := \underline{\ } \otimes \rho, \tag{142}$$

the chirality operator is given by

$$\chi_t := \gamma_5 \otimes \chi. \tag{143}$$

The definition of the generalized Dirac operator

$$\mathcal{D}_t := \partial\!\!\!/ \otimes 1 + \gamma_5 \otimes \mathcal{D} \tag{144}$$

is well motivated from differential geometry. We denote by δ_t the differential of the universal differential envelope $\hat{\Omega}\mathcal{A}_t$. This is only an auxiliary construction and we shall never need δ_t expressed in terms of d and δ. After passage to the quotient the differential, still denoted by δ_t, will be the concrete operator $-i\,[\mathcal{D}_t, \cdot\,]$, which we shall have to calculate in terms of $\partial\!\!\!/$ and $\mathcal{D}$. To alleviate notations we shall often omit the $\otimes$ and write e.g. $\delta_t\, fa$ for $\delta_t\,(f \otimes a)$. Again the good differential algebra $\Omega\mathcal{A}_t$ is obtained as a quotient via the homomorphism

$$\begin{aligned} &\pi_t(f_0 a_0\, \delta_t\, f_1 a_1 \ldots \delta_t\, f_p a_p) \\ &\quad := (-i)^p \rho_t(f_0 a_0)\left[\mathcal{D}_t, \rho_t(f_1 a_1)\right] \ldots \left[\mathcal{D}_t, \rho_t(f_p a_p)\right]. \end{aligned} \tag{145}$$

Our basic variable H_t is an antihermitian 1-form,

$$H_t \in \Omega^1 \mathcal{A}_t, \qquad H_t^* = -H_t. \tag{146}$$

Its curvature is the hermitian 2-form

$$C_t := \delta_t\, H_t + H_t^2 \tag{147}$$

used to calculate the functional

$$V_t(H_t) := (C_t, C_t)\,, \tag{148}$$

where the scalar product now involves the Dixmier trace in $\mathcal{S}$ and the trace in $\mathcal{H}$. The main miracle of Connes' recipe can be summarized in the following

Theorem 1. *We have the following decomposition:*

$$\begin{aligned} &H_t = A + H, \\ &\; A \in \Omega^1(M, \rho(\mathfrak{g})) \hookrightarrow \Omega^1\mathcal{F} \otimes \Omega^0\mathcal{A}, \\ &H^* = -H \in \Omega^0(M, \Omega^1\mathcal{A}) \cong \Omega^0\mathcal{F} \otimes \Omega^1\mathcal{A}, \end{aligned} \tag{149}$$

and

$$C_t = F + C - D\Phi\gamma_5\,, \tag{150}$$

with $\mathfrak{g}$ *the Lie algebra of the group of unitaries,*

$$\mathfrak{g} := \{X \in \mathcal{A}, X^* = -X\}\,, \tag{151}$$

with the field strength

$$F := \mathrm{d}A + A^2 = \mathrm{d}A + \tfrac{1}{2}[A, A] \quad \in \Omega^2(M, \rho(\mathfrak{g}))\,, \tag{152}$$

and the covariant derivative

$$D\Phi := \mathrm{d}\Phi + [A\Phi - \Phi A] \quad \in \Omega^1(M, \Omega^1\mathcal{A}). \tag{153}$$

Recall $\Phi := H - i\mathcal{D}$. Finally the generalized Higgs potential reads

$$V_t(A+H) = \int_M \mathrm{tr}(F * F) + \int_M \mathrm{tr}(D\Phi^* * D\Phi) + \int_M *V(H) - \int_M *V_0(H) \tag{154}$$

with

$$V_0(H) := \mathrm{tr}\left[(\alpha C)^* \alpha C\right], \tag{155}$$

and α is the linear map

$$\alpha : \Omega^2\mathcal{A} \longrightarrow \rho(\mathcal{A}) + \pi(\delta(\ker \pi)^1) \tag{156}$$

determined by the two equations

$$\mathrm{tr}\left[R^*(C - \alpha C)\right] = 0 \quad \text{for all } R \in \rho(\mathcal{A}), \tag{157}$$

$$\mathrm{tr}\left[K^* \alpha C\right] = 0 \quad \text{for all } K \in \pi(\delta(\ker \pi)^1). \tag{158}$$

The scalar product is the finite dimensional one of Section 1, the x dependence of C can be ignored.

We have two basic variables. A is a genuine gauge potential (non-abelian if $\mathcal{A}$ is non-commutative), that is, a differential 1-form on M with values in the Lie algebra $\mathfrak{g}$ of the group of unitaries of $\mathcal{A}$ represented on $\mathcal{H}$. H is as before, however, now with a differentiable x dependence, i.e., H is a multiplet of genuine scalar fields. The generalized Higgs potential reproduces the complete bosonic action of a Yang–Mills–Higgs model, namely the Yang–Mills action, the covariant Klein–Gordon action and the integral of the modified Higgs potential $V - V_0$. The modified potential is still a non-negative polynomial of fourth order in the scalar fields, in fact, as we shall see,

$$V - V_0 = \mathrm{tr}\left[(C - \alpha C)^*(C - \alpha C)\right]. \tag{159}$$

The entire action is gauge invariant. An element g of the group of unitaries of $\mathcal{A}_t$ is a differentiable function from spacetime into the finite dimensional group $G = \{g \in \mathcal{A}, gg^* = g^*g = 1\}$. Therefore these elements are genuine gauge transformations. Under the gauge group our fields transform as

$$A^g = \rho(g)A\rho(g^{-1}) + \rho(g)\mathrm{d}\rho(g^{-1}), \tag{160}$$

$$H^g = \rho(g)H\rho(g^{-1}) + \rho(g)\delta\rho(g^{-1}). \tag{161}$$

Before proving the theorem let us compute the modified Higgs potential V_0 for our example $\mathcal{A} = M_2(\mathbb{C}) \oplus \mathbb{C}$. Recall the generic elements

$$R=\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad a \in M_2(\mathbb{C}),\ b \in \mathbb{C}, \tag{162}$$

$$K=\begin{pmatrix} \Delta k & 0 \\ 0 & 0 \end{pmatrix}, \quad k \in M_2(\mathbb{C}), \tag{163}$$

$$C=\begin{pmatrix} 0 & 0 \\ 0 & M^*cM \end{pmatrix}, \quad c \in M_2(\mathbb{C}). \tag{164}$$

Therefore

$$\alpha C=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & |m|^2 c_{11} \end{pmatrix}, \tag{165}$$

$$V_0=\operatorname{tr}\left[(\alpha C)^2\right]=|m|^4(c_{11})^2=|m|^4(1-\phi_1^*\phi_1)^2, \tag{166}$$

and the modified Higgs potential vanishes; through the process of tensorizing we have lost the precious property of spontaneous symmetry breaking. However, this loss of symmetry breaking is particular to our example and not a typical feature. In fact a slight modification of the example, namely adding more families, remedies the evil. Consider the same algebra $\mathcal{A}=M_2(\mathbb{C})\oplus\mathbb{C}$ with a new Hilbert space consisting of N copies of the old one,

$$\mathcal{H}:=(\mathbb{C}^2\oplus\mathbb{C})\otimes\mathbb{C}^N. \tag{167}$$

E.g. for two families, $N=2$, the representation is by the 6×6 matrices

$$\rho(a,b):=\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & b \end{pmatrix} \tag{168}$$

if written with respect to the suggestive basis

$$(e_L,\nu_{eL},\mu_L,\nu_{\mu L},e_R,\mu_R). \tag{169}$$

The mass matrix

$$M=\begin{pmatrix} m \\ 0 \end{pmatrix} \tag{170}$$

involves now a non-degenerate, complex $N\times N$ matrix m, which should be thought of as mass matrix of the charged leptons. In other words, the basis is ordered differently here:

$$(e_L,\mu_L,\nu_{eL},\nu_{\mu L},\cdots),\quad N=2. \tag{171}$$

The formulas of Section 1 generalize naturally to the new situation,

$$MM^*=\begin{pmatrix} mm^* & 0 \\ 0 & 0 \end{pmatrix}=1\otimes\Delta+\sigma_3\otimes\Delta,\quad \Delta:=\tfrac{1}{2}mm^*. \tag{172}$$

The generic elements $R\in\rho(\mathcal{A})$, $K\in\pi(\delta(\ker\pi)^1)$ and $C\in\Omega^2\mathcal{A}$ become

$$R = \begin{pmatrix} a \otimes 1 & 0 \\ 0 & b \otimes 1 \end{pmatrix}, \quad a \in M_2(\mathbb{C}),\ b \in \mathbb{C}, \tag{173}$$

$$K = \begin{pmatrix} k \otimes \Delta & 0 \\ 0 & 0 \end{pmatrix}, \quad k \in M_2(\mathbb{C}), \tag{174}$$

$$C = \begin{pmatrix} 0 & 0 \\ 0 & M^*(c \otimes 1)M \end{pmatrix}. \tag{175}$$

A straightforward calculation yields

$$\alpha C = \begin{pmatrix} 0 & 0 \\ 0 & N^{-1}\,\mathrm{tr}(m^*m)c_{11} \otimes 1 \end{pmatrix} \tag{176}$$

and the (modified) Higgs potential

$$V - V_0 = \left[\mathrm{tr}(m^*m)^2 - \frac{1}{N}(\mathrm{tr}\ m^*m)^2\right](1 - \phi_1^*\phi_1)^2 \tag{177}$$

does break the symmetry spontaneously if m^*m has distinct eigenvalues.

Proof of the theorem. Our conventions are summarized in Table 1. In the following we compute only $\Omega^1\mathcal{A}_t$ and $\Omega^2\mathcal{A}_t$ for the particular case at hand. Details of the general case can be found in Ref. [2]. To get started, we need the commutator $[\mathcal{D}_t, \underline{f}\rho(a)]$. Using

$$\mathcal{D}_t = \begin{pmatrix} \partial\!\!\!/ & \gamma_5 M \\ \gamma_5 M^* & \partial\!\!\!/ \end{pmatrix} \tag{178}$$

we obtain

$$[\mathcal{D}_t, \underline{f}\rho(a)] = i\gamma(\mathrm{d}f)\rho(a) + \gamma_5\underline{f}\,[\mathcal{D}, \rho(a)]\,, \tag{179}$$

$$\pi_t(f_0 a_0\, \delta_t\, f_1 a_1) = \pi_D(f_0 \mathrm{d} f_1)\rho(a_0 a_1) + \underline{f_0 f_1}\gamma_5\pi(a_0\delta a_1). \tag{180}$$

Therefore $\Omega^1\mathcal{A}_t$ decomposes into a direct sum

$$\Omega^1\mathcal{A}_t = \Omega^1\mathcal{F} \otimes \rho(\mathcal{A}) \oplus \mathcal{F} \otimes \Omega^1\mathcal{A} \tag{181}$$

and its general element can be put under the form

$$H_t = A + H, \tag{182}$$

where

$$A = \pi_t(f_0 a\, \delta_t\, f_1 1) \quad \in \Omega^1\mathcal{F} \otimes \rho(\mathcal{A}) \tag{183}$$

is a 1-form on spacetime with values in $\rho(\mathcal{A})$ and

$$H = \pi_t(f a_0\, \delta_t\, 1 a_1) \quad \in \mathcal{F} \otimes \Omega^1\mathcal{A} \tag{184}$$

is an internal 1-form as in Section 1 but now with a differential x dependence. Later we shall impose that H_t be antihermitian, in particular that A take values in the Lie algebra $\mathfrak{g}$,

$$\mathfrak{g} := \{X \in \mathcal{A},\quad X^* = -X\} \tag{185}$$

Table 1
Conventions

algebra	$\mathcal{A} \ni a$	$\mathcal{F} \ni f : M \to \mathbb{C}$	$\mathcal{A}_t = \mathcal{F} \otimes \mathcal{A} \ni fa$
Hilbert space	$\mathcal{H} = \mathcal{H}_L \oplus \mathcal{H}_R$	$\mathcal{S} = L^2(M, \mathbb{C}^4) \ni \Psi$	$\mathcal{H}_t = \mathcal{S} \otimes \mathcal{H}$
chirality	χ	γ_5	$\chi_t = \gamma_5 \otimes \chi$
representation	$\rho(a) = \begin{pmatrix} \rho_L(a) & 0 \\ 0 & \rho_R(a) \end{pmatrix}$	$\underline{f}$	$\rho_t(fa) = \underline{f}\rho(a)$
"Dirac" operator	$\mathcal{D} = \begin{pmatrix} 0 & M \\ M^* & 0 \end{pmatrix}$	$\partial\!\!\!/$	$\mathcal{D}_t = \partial\!\!\!/ \otimes 1 + \gamma_5 \otimes \mathcal{D}$
universal differential envelope	$\hat{\Omega}\mathcal{A},\ \delta$	$\hat{\Omega}\mathcal{F},\ \mathrm{d}$	$\hat{\Omega}\mathcal{A}_t,\ \delta_t$
its scalar product	$\langle \cdot, \cdot \rangle = \mathrm{tr}[\,\cdot^* \cdot\,]$	$\langle \cdot, \cdot \rangle = \mathrm{tr}_\omega[\,\cdot^* \cdot\, \lvert\partial\!\!\!/\rvert^{-4}]$	$\langle \cdot, \cdot \rangle = \mathrm{tr}_\omega\, \mathrm{tr}[\,\cdot^* \cdot\, \lvert\partial\!\!\!/\rvert^{-4}]$
orthogonal projector	P	P_D	P_t
homomorphism	$\pi(a_0\delta a_1)$	$\pi_D(f_0 \mathrm{d} f_1)$	$\pi_t(f_0a_0\,\delta_t\,f_1a_1)$
the differential algebra	$\Omega\mathcal{A},\ \delta$	$\Omega\mathcal{F} \cong \Omega M,\ \mathrm{d}$	$\Omega\mathcal{A}_t,\ \delta_t$
its scalar product	$(\cdot, \cdot) = \langle \cdot, P \cdot \rangle$	$(\cdot, \cdot) = \langle \cdot, P_D \cdot \rangle$	$(\cdot, \cdot) = \langle \cdot, P_t \cdot \rangle$
Higgses, gauge potential	$H \in \Omega^1\mathcal{A},\ H^* = -H$	$A \in \Omega^1\mathcal{F},\ A^* = -A$	$H_t \in \Omega^1\mathcal{A}_t,\ H_t^* = -H_t$
curvature	$C = \delta H + H^2 \in \Omega^2\mathcal{A}$	$F = \mathrm{d}A + A^2 \in \Omega^2\mathcal{F}$	$C_t = \delta_t H_t + H_t^2 \in \Omega^2\mathcal{A}_t$
Higgs potential, action	$V(H) = (C, C)$	$S_{\mathrm{Maxwell}}(A) = (F, F)$	$V_t = (C_t, C_t)$

represented on $\mathcal{H}$, and H is antihermitian.

On level two we have

$$\begin{aligned}\pi_t(f_0a_0\,\delta_t\,f_1a_1\,\delta_t\,f_2a_2) = {}& \pi_D(f_0\mathrm{d}f_1\mathrm{d}f_2)\rho(a_0a_1a_2) + \underline{f_0f_1f_2}\pi(a_0\delta a_1\delta a_2) \\ & + i\gamma(f_0f_1\mathrm{d}f_2)\gamma_5\rho(a_0)\left[\mathcal{D}, \rho(a_1)\right]\rho(a_2) \\ & - i\gamma(f_0\mathrm{d}f_1f_2)\gamma_5\rho(a_0a_1)\left[\mathcal{D}, \rho(a_2)\right]. \end{aligned} \tag{186}$$

Consequently

$$\begin{aligned}\pi_t(\hat{\Omega}^2\mathcal{A}_t) = {}& \left[\pi_D(\hat{\Omega}^2\mathcal{F}) \otimes \rho(\mathcal{A}) + \mathcal{F} \otimes \pi(\hat{\Omega}^2\mathcal{A})\right] \\ & \oplus \pi_D(\hat{\Omega}^1\mathcal{F}) \otimes \pi(\hat{\Omega}^1\mathcal{A}). \end{aligned} \tag{187}$$

Remember from the last section that spacetime zero- and two-forms mix before division by the junk. This entails that the sum in the above bracket is not direct. Next we compute the tensor junk. Consider a general element of $\Omega^1\mathcal{A}_t$,

$$\pi_t(f_0a\,\delta_t\,f_1 1) + \pi_t(fa_0\,\delta_t\,1a_1). \tag{188}$$

Its pre-image p belongs to $(\ker \pi_t)^1$ if and only if $\pi_D(f_0\mathrm{d}f_1) = 0$ and $\pi(a_0\delta a_1) = 0$, in which case

$$\pi_t(\delta_t p) = \pi_D(\mathrm{d}f_0\mathrm{d}f_1)\rho(a) + \underline{f}\pi(\delta a_0\delta a_1). \tag{189}$$

It follows that

$$\pi_t(\,\delta_t\,(\ker\pi_t)^1) = \pi_D(\mathrm{d}(\ker\pi_D)^1) \otimes \rho(\mathcal{A}) + \mathcal{F} \otimes \pi(\delta(\ker\pi)^1)\,, \tag{190}$$

where again the sum is not direct and by division we get

$$\Omega^2\mathcal{A}_t = \left[\Omega^2\mathcal{F} \otimes \rho(\mathcal{A}) + \mathcal{F} \otimes \Omega^2\mathcal{A}\right] \oplus\ \Omega^1\mathcal{F} \otimes \Omega^1\mathcal{A}. \tag{191}$$

Our next task is to compute the total curvature $C_t = \delta_t H_t + H_t^2$ with $H_t = A + H$ and

$$A = \pi_t(f_0 a\,\delta_t\,f_1 1) = \pi_D(f_0\mathrm{d}f_1)\rho(a), \tag{192}$$

$$H = \pi_t(fa_0\,\delta_t\,1a_1) = \underline{f}\gamma_5\pi(a_0\delta a_1). \tag{193}$$

We have

$$\begin{aligned}\delta_t A &= \pi_t(\,\delta_t\,f_0 a\,\delta_t\,f_1 1) = \pi_D(\mathrm{d}f_0\mathrm{d}f_1)\rho(a) - \pi_D(f_0\mathrm{d}f_1)\gamma_5\pi(\delta a)\\ &=: \mathrm{d}A - \delta A\gamma_5,\end{aligned} \tag{194}$$

$$\begin{aligned}\delta_t H &= \pi_t(\,\delta_t\,fa_0\,\delta_t\,1a_1) = \underline{f}\pi(\delta a_0\delta a_1) + \pi_D(\mathrm{d}f)\gamma_5\pi(a_0\delta a_1)\\ &=: \delta\tilde{H} + \mathrm{d}\tilde{H}\gamma_5,\end{aligned} \tag{195}$$

$$\begin{aligned}(A+H)^2 &= \pi_D((f_0\mathrm{d}f_1)^2)\rho(a^2) + \underline{f}^2\pi((a_0\delta a_1)^2)\\ &\quad +\pi_D(f_0\mathrm{d}f_1 f)\gamma_5\pi(aa_0\delta a_1) - \pi_D(ff_0\mathrm{d}f_1)\gamma_5\pi(a_0\delta a_1 a)\\ &=:\ A^2 + \tilde{H}^2 + \left[A\tilde{H} - \tilde{H}A\right]\gamma_5.\end{aligned} \tag{196}$$

All together

$$\begin{aligned}C_t &= \mathrm{d}A - \delta A\gamma_5 + \delta\tilde{H} + \mathrm{d}\tilde{H}\gamma_5 + A^2 + \tilde{H}^2 + [A\tilde{H} - \tilde{H}A]\gamma_5\\ &= F + C + \left(i\,[\mathcal{D}, A] + \mathrm{d}\tilde{H} + [A\tilde{H} - \tilde{H}A]\right)\gamma_5\\ &= F + C - D\Phi\gamma_5\end{aligned} \tag{197}$$

with the curvatures

$$F := \mathrm{d}A + A^2, \qquad C := \delta\tilde{H} + \tilde{H}^2. \tag{198}$$

The covariant derivative

$$D\Phi := \mathrm{d}\Phi + [A\Phi - \Phi A] \quad \in \Omega\left(M, \Omega^1\mathcal{A}\right) \tag{199}$$

makes sense because of the homogeneous transformation law of $\Phi := \tilde{H} - i\mathcal{D}$. Note the purely algebraic term $\delta A = -i[\mathcal{D}, A]$ in the total curvature. This term generates the masses of the gauge bosons in the Lagrangian and does so by means of the fermionic mass matrix M in $\mathcal{D}$.

Finally we have to work out the scalar product in $\Omega^2\mathcal{A}_t$, the space where the curvatures live. As usual we start with the auxiliary scalar product $\langle\cdot,\cdot\rangle$ in $\pi_t(\hat{\Omega}\mathcal{A}_t)$ defined by

$$\langle \hat\phi_D \otimes \hat\phi, \hat\psi_D \otimes \hat\psi \rangle := \mathrm{tr}_\omega(\hat\phi_D^* \hat\psi_D |\not{\partial}|^{-4})\, \mathrm{tr}(\hat\phi^* \hat\psi),$$
$$\hat\phi_D, \hat\psi_D \in \pi_D(\hat\Omega\mathcal{F}),\ \hat\phi, \hat\psi \in \pi(\hat\Omega\mathcal{A}), \tag{200}$$

and define the scalar product on $\Omega\mathcal{A}_t$ by

$$(\phi_t, \psi_t) := \langle \phi_t, P_t \psi_t \rangle, \tag{201}$$

ϕ_t and ψ_t are any representatives of their classes and P_t is the orthogonal projector on $(\pi_t(\delta_t(\ker\pi_t)))^\perp$. For our purpose it is sufficient to know this projector in $\pi_t(\hat\Omega^2\mathcal{A}_t)$ where it is already non trivial, $P_t \neq P_D \otimes P$. Let us come back to the decomposition

$$\pi_t(\hat\Omega^2\mathcal{A}_t) = \left[\pi_D(\hat\Omega^2\mathcal{F}) \otimes \rho(\mathcal{A}) + \mathcal{F} \otimes \pi(\hat\Omega^2\mathcal{A})\right]$$
$$\oplus\, \pi_D(\hat\Omega^1\mathcal{F}) \otimes \pi(\hat\Omega^1\mathcal{A}) \tag{202}$$

with general elements in the three subspaces

$$\pi_t(f_0 a_0\, \delta_t\, f_1 1\, \delta_t\, f_2 1) = \pi_D(f_0 \mathrm{d} f_1 \mathrm{d} f_2)\rho(a_0), \tag{203}$$
$$\pi_t(f_0 a_0\, \delta_t\, 1 a_1\, \delta_t\, 1 a_2) = \underline{f}_0 \pi(a_0 \delta a_1 \delta a_2), \tag{204}$$
$$\pi_t(f_0 a_0\, \delta_t\, 1 a_1\, \delta_t\, f_2 1) = -\pi_D(f_0 \mathrm{d} f_2)\gamma_5 \pi(a_0 \delta a_1). \tag{205}$$

Our first conclusion is that the above direct sum is also an orthogonal sum because the trace in Eq. (127) is over γ_5 multiplied by an odd number of proper gamma matrices and vanishes. Therefore by Eq. (190) P_t leaves the third subspace untouched and we concentrate on the restriction P_1 of P_t to the first two subspaces. Let us introduce the following short hands:

$$U := \pi_D(\hat\Omega^2\mathcal{F}) \otimes \rho(\mathcal{A}), \tag{206}$$
$$U_0 := \pi_D(\mathrm{d}(\ker\pi_D)^1) \otimes \rho(\mathcal{A}) \cong \mathcal{F} \otimes \rho(\mathcal{A}) \tag{207}$$

and $U_\perp$ is the ortho-complement of U_0 in U,

$$U_\perp := (U_0)^{\perp U} \cong \Omega^2 M \otimes \rho(\mathcal{A}). \tag{208}$$

Likewise

$$W := \mathcal{F} \otimes \pi(\hat\Omega^2\mathcal{A}), \tag{209}$$
$$W_0 := \mathcal{F} \otimes \pi(\delta(\ker\pi)^1) \tag{210}$$

and $W_\perp$ is the ortho-complement of W_0 in W,

$$W_\perp := (W_\perp)^{\perp W} = \mathcal{F} \otimes \Omega^2\mathcal{A} \tag{211}$$

and P_1 is the orthogonal projector in $U + W$ on $(U_0 + W_0)^\perp \neq U_0^\perp + W_0^\perp$. Next we remark that $U_\perp$ is orthogonal to the other three subspaces,

$$U_\perp \cap (U_0 + (W_0 \oplus W_\perp)) = 0, \tag{212}$$
$$U_\perp \perp (U_0 + (W_0 \oplus W_\perp)) \tag{213}$$

because $\mathrm{tr}_4[\gamma^\mu\gamma^\nu] = \mathrm{tr}_4[\gamma^\nu\gamma^\mu]$. This means that also $V_\perp$ decouples and it remains to compute the restriction P_2 of P_t to

$$U_0 + (W_0 \oplus W_\perp) = \mathcal{F} \otimes \left[\rho(\mathcal{A}) + (\pi(\delta(\ker\pi)^1) \oplus \Omega^2\mathcal{A})\right]. \tag{214}$$

In this space now the x dependence is trivial and can be ignored reducing the calculation to a finite dimensional one in $\left[\rho(\mathcal{A}) + (\pi(\delta(\ker\pi)^1) \oplus \Omega^2\mathcal{A})\right]$ with scalar product defined by tr, the trace over $\mathcal{H}$. What we need is P_2C, $C \in \Omega^2\mathcal{A}$. Since P_2 is an orthogonal projector onto $(U_0 + W_0)^\perp$, $C - P_2C$ is perpendicular to $(U_0 + W_0)^\perp$,

$$C - P_2C =: \alpha C \ \in (U_0 + W_0)^{\perp\perp} = U_0 + W_0 \tag{215}$$

or

$$P_2C = C - \alpha C. \tag{216}$$

Finally by Pythagoras

$$\begin{aligned}\langle C, P_tC\rangle = \langle C, P_2C\rangle = \langle C, C - \alpha C\rangle &= (C, C) - \langle C, \alpha C\rangle \\ &= (C, C) - \langle P_2C + \alpha C, \alpha C\rangle \\ &= (C, C) - \langle \alpha C, \alpha C\rangle.\end{aligned} \tag{217}$$

4. Remarks

Instead of conclusions we offer a list of remarks.

The standard model

The first question is of course: can the standard model be accommodated in Connes' approach? The answer is yes, but this requires three new items to be added [1,3,4] to Connes' model building kit as reviewed so far. In our example the group of unitaries, $U(2) \times U(1)$, is too big by one $U(1)$ factor to describe electroweak interactions and there are two complex Higgs doublets, one too many. Both points are cured readily by replacing the internal algebra by $\mathcal{A} = \mathbb{H} \oplus \mathbb{C}$, $\mathbb{H}$ being the quaternions, keeping the K-cycle unchanged. Straightening out the hypercharges requires a first new item, the so called unimodularity condition, which reduces the group of unitaries by a purely algebraic restriction. But then one has to start from a bigger algebra, which is anyhow needed to include strong interactions. Since quantum chromodynamics is a vector like theory its inclusion is easy except for two points. First its group, $SU(3)$, is the group of unitaries of no algebra $\mathcal{A}$. This problem is taken care of by the unimodularity condition. Second the quarks come in a representation, that is a tensor product, $(3, 2)$, and therefore not available so far, a problem solved by introducing bimodules, the second item. As explained in our example the Higgs and gauge boson masses are determined by the fermion mass matrix. Also it is obvious that the coupling constants in the bosonic action V_t are related. In the complete standard model à la Connes these relations are [4,5] $m_W = \frac{1}{2}\, m_t$, $m_H = \frac{1}{2}\sqrt{69/7}\, m_t$, with all fermion masses neglected against the top mass, and $g_3 = g_2$, $\sin^2\theta_W = 3/8$. Of course these classical relations are unstable under quantum corrections [6] and there are at least two possible attitudes with respect to this dilemma. The first says we do not know yet what quantum field theory is in this new context. The other attitude is the third item: The trace in the Hilbert space $\mathcal{H}$ that we have used to define the scalar product is not unique (up to normalisation) and taking the most general traces produces the standard model without the constraints on masses and coupling constants.

Other generalisations

We have only considered a particular case of Connes' algorithm. In general the total algebra $\mathcal{A}_t$ is not a tensor product of $\mathcal{F}$ and a finite dimensional algebra $\mathcal{A}$. Also the basic variable H_t can live in a more general space than $\Omega^1\mathcal{A}_t$, it can be a connection on any hermitian finite projective module over $\mathcal{A}_t$. These generalisations do not seem to help with the above mentioned problems.

Other algebras

Up to now only few algebras $\mathcal{A}$ have been explored besides the standard model. It is most intriguing that the simplest typical example we know so far is already quite involved and reproduces most features of electroweak interactions. The other algebras considered in this context are $M_5(\mathbb{C})$ [7] and Cliff(10) [8] in order to reproduce the $SU(5)$ and $SO(10)$ grand unified theories. The first one fails because the fermions in the $SU(5)$ model sit in a $\bar{5}+10$, the 10 comes from a tensor product and does not fit Connes' rules. The second model is left-right symmetric and it is difficult to obtain the complicated Higgs sector necessary in $SO(10)$. Finally a smaller left-right symmetric model with gauge group $U(2)\times U(2)$ has been worked out [9]. As in the $SO(10)$ model, the gauge symmetry is broken spontaneously and parity remains unbroken.

Splinters

It does not look easy to get rid of imaginary time. In non compact, pseudo Riemannian spacetime traces and scalar products are ill defined. The Dirac operator has continuous spectrum, action integrals diverge and there is certainly more to do than simply invoking Wick rotation. There is no convincing motivation for the time integral in $\mathcal{S} = L^2(M, \mathbb{C}^4)$.

Other non commutative schemes

Connes' algorithm to produce a differential algebra $\Omega\mathcal{A}$ starting from an algebra $\mathcal{A}$—such that the algebra $\mathcal{F}$ of functions on a manifold M reproduces de Rham's differential algebra ΩM—is not unique. In fact already in 1988 Dubois-Violette [10] introduced a different such algorithm using derivations. This algorithm is successfully used by the southern Paris group to obtain particle models [11]. These models share the attractive features of Connes' theory, they also unify gauge and Higgs bosons. In their scheme however the Higgs bosons transforms according to the adjoint representation of the group of unitaries irrespective of fermion representations. Their scheme has been generalized by Balakrishna, Gürsey and Wali [12] to include other Higgs representations. Another approach due to Coquereaux [13] takes immediately the differential algebra $\Omega\mathcal{A}_t$ as starting point. The approach is thereby more transparent and less rigid. Many of its physical features have been worked out by the Marseille–Mainz group [14].

Non-commutative algebras and quantum physics

The non-commutative models are clearly inspired by the mathematics of quantum mechanics, operator algebras. There are two main differences between non commutative and quantum. Ad one, the passage from classical to quantum mechanics can be considered as replacing the commutative algebra of functions on phase space, the "observables", by a non-commutative algebra [15]. In the above models not phase space but space time is rendered non-commutative. Ad two, in quantum mechanics the non-commutative algebra is God given and contains a dimensionful parameter, Planck's constant. A nice example illustrating the interplay of quantum mechanics and non-commutative geometry is given by Madore [16]. We repeat that a generalisation of quantum field theory to the non-commutative setting is still lacking.

Parallel universes

The first example studied by Connes and Lott [1] was $\mathcal{A} = \mathbb{C} + \mathbb{C}$, $\mathcal{H} = \mathbb{C} + \mathbb{C}$. It has a nice geometric interpretation in terms of a Riemannian manifold $M = M_L + M_R$, disjoint union of two identical Riemannian manifolds M_L and M_R separated by the constant distance $(|m_1|^2 + |m_2|^2)^{-1/2}$. The left handed fermions live on M_L, the right handed fermions on M_R. This model can also be interpreted as a Kaluza–Klein theory with a discrete fifth dimension consisting of two points. The Kaluza–Klein analogy is present in all non-commutative models. It has been worked out in detail by the southern Paris group [17] and served as initial motivation for Coquereaux's scheme [13].

Gravity

A question of fundamental importance is: can the Einstein–Hilbert action be fit into the non commutative frame? Again the answer is not unique. A first proposal is due to the Zürich group [18]. Starting from Einstein–Cartan's theory they arrive at a tensor scalar theory. The scalar has a geometric interpretation as the now variable distance between parallel universes. In a recent paper Chamseddine and Fröhlich [19] have coupled this scalar to the standard model and after addition of some effective Coleman–Weinberg potential they obtain a striking prediction for the top and Higgs masses,

$$146.2 \leq m_t \leq 147.4 \text{ GeV}, \qquad 117.3 \leq m_H \leq 142.6 \text{ GeV}. \tag{218}$$

Connes [20] has found a more intrinsic way to incorporate gravity: he computes the Dixmier trace or more properly the Wodizicki residue of the (true) Dirac operator to the power minus two and obtains the Einstein–Hilbert action.

Unification

Grand unification was based on the attractive idea to replace a direct product of groups by a simple group. In the same spirit and in order to reconcile particle interactions and gravity it seems attractive to look for an algebra $\mathcal{A}_t$, that is not just a tensor product of the algebra $\mathcal{F}$ of functions on spacetime and an internal algebra, but that is still sufficiently close to $\mathcal{F}$ to allow spacetime to subsist in some form. A 2-dimensional example of such an algebra is the non-commutative torus which plays an intriguing role in solid state physics.

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